

Optimal execution comparison across risks and dynamics, with solutions for displaced diffusions^{*}

Damiano Brigo
 Dept. of Mathematics
 Imperial College London
 and Capco Research Institute
 damiano.brigo@imperial.ac.uk

Giuseppe Di Graziano
 Dept. of Mathematics
 King's College London
 and Deutsche Bank, London
 giuseppe.di-graziano@db.com

First Version: April 10, 2013.

Abstract

We solve the optimal trade execution problem in the Almgren and Chirss framework with the Value-at-risk / Expected Shortfall based criterion of Gatheral and Schied when the underlying unaffected stock price follows a displaced diffusion model. The displaced diffusion model can conveniently model at the same time situations where either an arithmetic Brownian motion (ABM) or a geometric Brownian motion (GBM) type dynamics may prevail, thus serving as a bridge between the ABM and GBM frameworks. We introduce alternative risk criteria and we notice that the optimal trade execution strategy little depends on the specific risk criterion we adopt. In most situations the solution is close to the simple Volume Weighted Average Price (VWAP) solution regardless of the specific diffusion dynamics or risk criterion that is chosen, especially on realistic trading horizons of a few days or hours. This suggests that more general dynamics need to be considered, and possibly more extreme risk criteria, in order to find a relevant impact on the optimal strategy.

AMS Classification Codes: 60H10, 60J60, 91B70;

JEL Classification Codes: C51, G12, G13

Keywords: Optimal trade execution, Algorithmic trading, Displaced Diffusion, HJB equation, calculus of variations, risk measures, Value at Risk, Expected Shortfall, Squared-Asset Expectation, Market Impact.

^{*}We are grateful to Alexander Schied for helpful correspondence on his invariance result. This work expresses the opinion of its authors and is in no way representing the opinion of the institutions the authors work for

1 Introduction

A fundamental problem for algorithmic traders is how to partition a large trade into smaller trades so as to minimize the effect of market impact. In a number of seminal papers including [1] and [2], Almgren and Chriss formalized this problem by combining expected execution cost and risk. The framework in [1] and [2] has linearly increasing execution costs (in the trading rate) whereas the risk criterion is given by the variance. The advantage of such a setup is the possibility to find a closed-form analytical solution to the optimal execution problem. This solution is static and is usually in the class of the Volume Weighted Average Price (VWAP) solutions. A disadvantage of this framework is that it needs to assume an extremely stylized dynamics for the underlying unaffected stock, namely an arithmetic Brownian motion (ABM), possibly leading to arbitrarily negative asset values. Another consequence of the framework is that adaptive trade schedules formulated on the current stock price level are suboptimal.

Gatheral and Schied [3] solve the same problem under the more realistic assumption that the unaffected underlying stock follows a geometric Brownian motion (GBM). Besides being the cornerstone of modern basic option pricing theory, GBM has the clear advantage of implying positive stock prices in all scenarios. However, under the GBM assumption with variance as risk criterion, the optimal execution problem is not tractable. The problem has been studied numerically in [4] and [5], where it is found that under GBM the optimal trading rate increases with the stock price and that the cost-risk efficient frontier is almost the same under ABM and GBM, implying that the static solution obtained by solving the ABM version is not very suboptimal even under the different GBM assumption.

However, Gatheral and Schied look for an analytic solution rather than a numerical one. This leads them to resorting to a different risk criterion, namely the Value at Risk measure. This is adopted in [3] and the problem is solved analytically. An explicit closed form expression for the trading rate is obtained. The authors are then able to analyze precisely the optimal strategy under the GBM assumption, and can explicitly compare the optimal strategy under GBM with the optimal strategy for the risk criterion under ABM.

In this paper we extend the Gatheral and Schied [3] analytic result to a more general dynamics, namely the displaced diffusion (DD) model. This dynamics is more general than both ABM and GBM, but comprises behaviours very similar to either ABM or GBM for different values of the parameters in the dynamics itself. In particular, it allows to model volatility as a combination of constant absolute volatility (as in the ABM) and level-proportional absolute volatility (as in GBM).

This is a step forward in that it can be seen as a model that unifies the Almgren and Chriss and Gatheral and Schied frameworks into a single tractable setup.

We further introduce a risk criterion we call SAE (Squared-Asset Expectation) to investigate how a different risk criterion would change the optimal solution. We solve the related problem through an approximation related to calculus of variations and we notice that the optimal trade execution strategy seems to depend little on the specific risk criterion we adopt. In particular, we find that in most situations the solution is close to the Volume Weighted Average Price (VWAP) solution regardless of the specific diffusion dynamics or risk criterion that is chosen. This holds unless extreme and unrealistic assumptions are considered either for the dynamics parameters or for the risk criterion parameters. This leads to suggest that more general dynamics need to be considered, and possibly more extreme risk criteria, in order to find a relevant difference in the optimal strategy.

The paper is organized as follows: Section 2 introduces the optimal trade execution problems with a few examples of stylized dynamics for the unaffected asset price, and with simple market impact and cost assumptions. In addition, risk functions are introduced and Value at Risk (VaR) and Expected Shortfall (ES) are discussed in detail. Section 3 introduces the optimal solution for the case where the dynamics is a displaced diffusion and where the risk function is either VaR or ES. Section 4 explains how the DD with VaR/ES criteria escapes Schied's invariance result. Section 5 introduces a different Squared-Asset Expectation (SAE) risk criterion that is closer to the variance, and derives approximate solutions of the execution problem under that criterion. Section 6 compares the optimal execution solutions under different criteria and dynamics, with numerical examples and discussion, whereas Section 7 concludes the paper.

2 The trade execution problem under different dynamics

Following [1], [2] and especially [3] we assume that the trader's number of shares follows an absolutely continuous trajectory

$$t \mapsto x(t).$$

Given this trading path, the price at which the transaction occurs is given by

$$\tilde{S}_t = S_t + \eta \dot{x}(t) + \gamma(x(t) - x_0)$$

where η and γ are constants and S is the process for the unaffected stock price level. We can identify the following impact terms:

- $\eta \dot{x}(t)$ is the instantaneous impact of trading $dx(t) = \dot{x}(t)dt$ shares in $[t, t + dt)$ and only affects the $[t, t + dt)$ order.
- $\gamma(x(t) - x_0)$ represents the permanent impact price that has been accumulating over $[0, t]$ by all transactions up to t .

2.1 ABM and GBM dynamics for the unaffected stock S

The papers [1] and [2] assume that the unaffected (forward) stock price process S follows an ABM:

$$dS_t = \sigma S_0 dW_t, \quad S_0 \quad (\text{ABM})$$

where W is a standard Brownian motion.

Gatheral and Schied [3] assume instead that

$$dS_t = \sigma S_t dW_t, \quad S_0 \quad (\text{GBM}).$$

The absolute volatility in the ABM model is supposed to be proportional to S_0 for dimensionality issues and for comparison with GBM. However the key fact is that the absolute volatility in ABM is constant and equal to $\bar{\sigma} = \sigma S_0$ whereas in GBM the absolute volatility σS_t is level proportional.

One can easily compute the probability \mathbb{P} that the asset is negative. One obtains

$$\mathbb{P}(S_t < 0) = \Phi(-(\sigma\sqrt{t})^{-1}) \quad (\text{ABM}), \quad \mathbb{P}(S_t < 0) = 0 \quad (\text{GBM})$$

Clearly, ABM has the problem, especially for long times, that the unaffected stock probability of going negative is large. In the limit for large t such probability becomes $1/2$. Luckily, trade execution is usually considered on short time scales, so that this problem is not as bad. However, GBM is a safe improvement since it immediately excludes negative stock values completely.

2.2 DD dynamics for the unaffected stock S

Consider now the DD model. We formulate such model for the unaffected stock price S as follows: given a GBM Y_t with volatility parameter σ ,

$$dY_t = \sigma Y_t dW_t, \quad Y_0 = S_0 - K$$

we define S as

$$S_t = K + Y_t \text{ or equivalently } dS_t = \sigma(S_t - K)dW_t, S_0 \text{ (DD)}$$

In other terms, a displaced diffusion model for S means that S is a shifted GBM. The shift is a constant K and the GBM is Y_t . To keep the asset positive, we clearly would need to assume $K \geq 0$.

If $K < 0$, the probability of a negative stock S can be computed as

$$\mathbb{P}(S_t < 0) = \Phi\left(\frac{-\ln(1 - S_0/K) + \sigma^2 t/2}{\sigma\sqrt{t}}\right) \text{ (DD with } K < 0\text{)}$$

One can clearly see that the probability of having a negative stock in the long run, when the shift is negative, tends to 1, so that for negative shifts the asymptotic probability of negative states is worse than for (ABM).

However, in general the DD model has the advantage of mimicking the key features in the GBM and ABM in a single model. Indeed, consider the absolute volatility in the DD model:

$$\sigma(S_t - K) = \sigma S_t(1 - K/S_t)$$

For large stock values S , the volatility looks like the level-proportional absolute volatility in a GBM model. Indeed, for large S the ratio K/S becomes small and one has

$$\sigma S_t(1 - K/S_t) \approx \sigma S_t \text{ for } S_t \gg K.$$

Instead, for small values of S , the absolute volatility looks more similar to the constant absolute volatility of the ABM, in that

$$\sigma(S_t - K) \approx \sigma(-K) \text{ for } S_t \approx 0.$$

This is usually summarized by practitioners by saying that "DD behaves like ABM for small stock values and as GBM for large ones".

2.3 The trade execution problem

We consider a trade execution strategy, described by the above absolutely continuous stochastic process

$$t \mapsto x(t)$$

representing the amount of shares that are left to be sold at time T . We assume that the process is adapted. Our aim is completing the sale of X shares by time T , so that our boundary conditions are

$$x(0) = X, \quad x(T) = 0.$$

The trade execution problem consists in finding the x as above that minimizes an objective function based on costs and risk terms. We look at those terms now.

2.4 Cost Function

The instantaneous cost of the strategy is given by the cost of buying $dx(t) = \dot{x}(t)dt$ shares at time $[t, t + dt)$ at the impacted stock price \tilde{S}_t , spending $\tilde{S}_t \dot{x}(t)dt$.

We can then write the costs arising from the strategy, following Gatheral and Schied [3] as

$$\begin{aligned} C(x) &:= \int_0^T \tilde{S}_t \dot{x}(t)dt = \int_0^T [S_t + \eta \dot{x}(t) + \gamma(x(t) - x_0)] \dot{x}(t)dt \\ &= -XS_0 - \int_0^T x(t)dS_t + \eta \int_0^T (\dot{x}(t))^2 dt + \gamma X^2/2 \end{aligned}$$

where we have used an integration by parts. This calculation does not involve the specific dynamics of S yet and is general.

2.5 Risk Function: Value at Risk

A common measure of risk is Variance. However, Gatheral and Schied [3] argue that for GBM a different interesting choice is a Value at Risk (VaR) risk based function. VaR is basically a given quantile of the difference between the value of the position at the time of calculation and the future value of the position at the risk horizon. It is a quantile of the future loss associated with the position. Let $\nu_{\alpha,t,h}$ be the Value at Risk measure computed at time t , for the position, for a given confidence level α (for example $\alpha = 0.95$ or $\alpha = 0.99$) and over a time horizon h . We may omit some arguments when clear from the context. This means that by definition, given the market filtration \mathcal{F} and given the probability measure \mathbb{P} , we have

$$\mathbb{P}\{S_t - S_{t+h} \leq \nu_{\alpha,t,h} | \mathcal{F}_t\} = \alpha.$$

If at time t we have $x(t)$ shares with price S_t , the time t VaR measure for a risk horizon h for that $x(t)$ position under the DD dynamics at confidence level α would be

$$\begin{aligned}\nu_t[x(t)(S_t - S_{t+h})] &= x(t)\nu_t[(S_t - S_{t+h})] = x(t)\nu_t[(Y_t + K - Y_{t+h} - K)] \\ &= x(t)\nu_t[(Y_t - Y_{t+h})] = x(t)\nu_t[Y_t(1 - \exp(-\sigma^2 h/2 + \sigma(W_{t+h} - W_t)))] \\ &= x(t)Y_t\nu_t[1 - \exp(-\sigma^2 h/2 + \sigma\sqrt{h}\epsilon)] = x(t)Y_tq_\alpha[1 - \exp(-\sigma^2 h/2 + \sigma\sqrt{h}\epsilon)]\end{aligned}$$

where ϵ is a standard normal random variable, where we have used the homogeneity property of VaR, and where in general $q_\alpha(X)$ is the α quantile of the distribution of the random variable X . Define the constant

$$\tilde{\lambda}_\alpha := q_\alpha[1 - \exp(-\sigma^2 h/2 + \sigma\sqrt{h}\epsilon)]$$

to obtain that

$$\nu_t[x(t)(S_t - S_{t+h})] = \tilde{\lambda}_\alpha x(t)Y_t = \tilde{\lambda}_\alpha x(t)(S_t - K).$$

This is the VaR measure for the instantaneous position at time t . If we average VaR over the life of the strategy we obtain the risk criterion

$$R^{\text{VaR}_\alpha}(x) := \tilde{\lambda} \int_0^T x(t)(S_t - K)dt.$$

This will be one of our measures of risk, to be included in the criterion to be minimized for optimality of the execution strategy.

2.6 Risk Function: Expected Shortfall

Expected shortfall $\mu_{\alpha,t,h}$ at time t for the horizon $t+h$ is defined as the expected value of the loss of the position beyond the VaR level, after the given time horizon, and for a given confidence level (the one used for VaR, say α). Hence it is defined as

$$\mu_{\alpha,t,h} := \mathbb{E}\{S_t - S_{t+h} | S_t - S_{t+h} \geq \nu_{\alpha,t,h}, \mathcal{F}_t\}.$$

Expected Shortfall is usually preferred to VaR on the basis of the fact that it inspects the whole tail beyond the chosen quantile. Moreover, expected shortfall is a coherent risk measure, namely it is sub-additive, while Value at Risk is not.

By proceeding analogously with the VaR case, if at time t we have $x(t)$ shares with price S_t , the time t expected shortfall measure for a risk horizon h for that $x(t)$ position under the DD dynamics at confidence level α would be

$$\mu_t[x(t)(S_t - S_{t+h})] = x(t)Y_t m_\alpha[1 - \exp(-\sigma^2 h/2 + \sigma\sqrt{h}\epsilon)]$$

where ϵ is a standard normal random variable, where we have used the homogeneity property of expected shortfall, and where in general $m_\alpha(X)$ is the expected shortfall for a loss given by the random variable X beyond the α quantile of the distribution of X . Through straightforward calculations one obtains

$$m_\alpha = \frac{1}{1-\alpha} \left[\Phi \left(\frac{\ln(1 - \tilde{\lambda}_\alpha) + \sigma^2 h/2}{\sigma\sqrt{h}} \right) - \Phi \left(\frac{\ln(1 - \tilde{\lambda}_\alpha) - \sigma^2 h/2}{\sigma\sqrt{h}} \right) \right] =: \hat{\lambda}_\alpha$$

and hence

$$\mu_t[x(t)(S_t - S_{t+h})] = x(t)Y_t \hat{\lambda}_\alpha = \hat{\lambda}_\alpha x(t)(S_t - K).$$

This is the expected shortfall measure for the instantaneous position at time t . If we average this measure over the life of the strategy we obtain the risk criterion

$$R^{\text{ES}_\alpha}(x) := \hat{\lambda} \int_0^T x(t)(S_t - K)dt.$$

This will be one of our measures of risk, to be included in the criterion to be minimized for optimality of the execution strategy.

2.7 Cost plus Risk

We are now ready to define our criterion to be minimized. We put together costs and risks in a single criterion

$$\mathbb{E}[C(x) + R(x)] = \mathbb{E} \left[C(x) + L\bar{\lambda} \int_0^T x(t)(S_t - K)dt \right].$$

In this criterion:

- L is a cost/risk leverage parameter that measures risk aversion in executing the order. The larger L , the more important the risk component of the criterion compared to the cost component;

- $\bar{\lambda}$ is either $\tilde{\lambda}$ or $\hat{\lambda}$ depending on whether we are using the VaR or expected shortfall risk function, leading to

$$\begin{aligned}\mathbb{E}[C(x) + R(x)] &= -XS_0 + \gamma X^2/2 + \eta \mathbb{E} \left[\int_0^T \dot{x}(t)^2 dt + L \frac{\bar{\lambda}}{\eta} \int_0^T x(t)(S_t - K) dt \right] \\ &= -XS_0 + \gamma X^2/2 + \eta \mathbb{E} \left[\int_0^T \dot{x}(t)^2 dt + L \check{\lambda} \int_0^T x(t) Y_t dt \right]\end{aligned}$$

where we have set $\check{\lambda} = \bar{\lambda}/\eta$.

The problem is now finding

$$x^* = \operatorname{arginf}_x \mathbb{E} \left[\int_0^T \dot{x}(t)^2 dt + L \check{\lambda} \int_0^T x(t) Y_t dt \right]. \quad (2.1)$$

3 Solution of the trade execution problem for a displaced diffusion

The above problem (2.1) has been solved by Gatheral and Schied [3]. If we pretend for a moment that Y is our true unaffected underlying stock, the criterion we have is the same as the criterion for a GBM Y , and this has been solved in [3]. Indeed, Theorem 1 in Gatheral and Schied [3] implies that the unique optimal trade execution strategy for the problem (2.1) above is

$$x_t^* = \frac{T-t}{T} \left[X - L \check{\lambda} \frac{T}{4} \int_0^t Y_u du \right]$$

with the minimized value being given by

$$\mathbb{E} \left[\int_0^T \dot{x}^*(t)^2 dt + L \check{\lambda} \int_0^T x^*(t) Y_t dt \right] = \frac{X^2}{T} + \frac{\check{\lambda} L T X Y_0}{2} - \frac{(L \check{\lambda})^2}{8\sigma^6} Y_0^2 \left(e^{\sigma^2 T} - 1 - \sigma^2 T - \frac{\sigma^4 T^2}{2} \right)$$

which, replacing Y by $S - K$, leads to the following

Theorem 3.1. (Optimal execution strategy for a displaced diffusion). *The unique optimal trade execution strategy attaining the infimum in (2.1) is*

$$x_t^* = \frac{T-t}{T} \left[X - \check{\lambda} L \frac{T}{4} \int_0^t (S_u - K) du \right]$$

Furthermore, the value of the minimization problem in (2.1) is given by

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \dot{x}^*(t)^2 dt + \check{\lambda} L \int_0^T x^*(t)(S_t - K) dt \right] \\ &= \frac{X^2}{T} + \frac{L\check{\lambda}TX(S_0 - K)}{2} - \frac{(L\check{\lambda})^2}{8\sigma^6}(S_0 - K)^2 \left(e^{\sigma^2 T} - 1 - \sigma^2 T - \frac{\sigma^4 T^2}{2} \right) \end{aligned}$$

4 DD under VaR or ES escapes Schied's invariance result

A quite interesting invariance result due to Alexander Schied is proven in [6]. The result is rather general, and here we are concerned with a special case. The paper [6] considers the risk criterion

$$R^{AS}(x) := \lambda \int_0^T x(t) S_t dt$$

which would be a VaR or ES type risk function when the unaffected price S follows a geometric brownian motion (as proven in [3]).

Then the paper [6] shows that as long as S follows a martingale, the optimal trade execution strategy x^* does not depend on the specific martingale dynamics one chooses. Since the displaced diffusion model we consider in this paper is also a martingale, one does then deduce that under the R^{AS} risk criterion and DD dynamics we obtain the same solution we would obtain under a GBM dynamics.

This seems to contradict our result in this paper, given that our solution is different from the GBM one (as we have K in our solution). In fact, however, there is no contradiction.

Indeed, we use a different risk criterion, since we use

$$\lambda \int_0^T x(t)(S_t - K) dt \quad \text{instead of} \quad \lambda \int_0^T x(t) S_t dt.$$

In the specific case of DD, the first criterion corresponds to the VaR or Expected Shortfall (depending on the choice of λ) of the position, whereas R^{AS} corresponds neither to VaR nor to ES. Hence if we include K in our criterion we are using a risk criterion that is different from [6] and this is why we escape the invariance result and our optimal solution is different from the GBM one.

We believe it is good to retain K in the criterion for the DD case because, in this way, we are really using VaR or expected shortfall as a risk criterion.

5 Optimal solution for displaced diffusion processes under an alternative square risk function

In order to test the robustness of the optimal strategy, we introduce the alternative "squared-asset expectation" (SAE) risk criteria

$$R^{SAE}(x) \equiv \lambda \int_0^T x^2(t) \sigma^2 E[S_t^2] dt. \quad (5.1)$$

This goes a little back towards considering the variance as a measure of risk.

Remark 5.1. (On the meaning of λ). *It is important to notice that while in the VaR and expected shortfall λ had a precise value emerging endogenously from the chosen criterion, here we have just introduced λ exogenously, as a parameter we are free to set in our criterion.*

The criterion above is similar to the one used by Almgren and Chriss in [2] but with the displaced diffusion replacing the arithmetic Brownian motion (ABM) as far as the dynamics of S_t are concerned and the expected value of the square of the price process replacing the square of the price process itself. Note that the term

$$g(t) \equiv E[S_t^2]$$

is increasing in t , whereas the corresponding term in [2] is constant and equal to S_0^2 . Moreover in the Almgren and Chriss framework the volatility of the asset $\tilde{\sigma}$ is expressed in units of the reference currency, whereas in this paper σ is a percentage number. Using the notation of this paper, the risk criteria employed in [2] can be re-written as

$$R^{AC} \equiv \lambda \int_0^T x^2(t) \sigma^2 S_0^2 dt. \quad (5.2)$$

Everything else being equal, the magnitude of the risk component (5.1) is higher as time goes by compared to (5.2). Finally note that the expected cost is the same whether S_t is modelled by a DD or an ABM, so the optimal solution depends only of the choice of risk function.

The optimal execution strategy can be derived by solving the optimisation problem

$$x^* = \operatorname{arginf}_x \mathbb{E} \left[\int_0^T \dot{x}(t)^2 dt + L \lambda \int_0^T \sigma^2 x^2(t) g(t) dt \right]. \quad (5.3)$$

Using calculus of variation, we show in appendix A that the optimal solution needs to satisfy the ODE

$$\ddot{x}(t) = k^2 g(t) x(t), \quad (5.4)$$

where

$$k \equiv \sigma \sqrt{L\lambda},$$

and the initial and terminal conditions are given by $x(0) = X$ and $x(T) = 0$ respectively. A solution to the boundary value problem above could be found using standard numerical routines. However, in appendix B we provide a simple and efficient alternative algorithm based on series expansions.

6 Comparing Risk Criteria

If we are to compare optimal strategies under the SAE risk criterion with the VaR and expected shortfall criteria strategies, we need to set λ in the SAE criterion in a way that makes the comparison sensible. We call this procedure "equalizing the λ 's".

A first possible way to equalize λ 's is to match the initial and final values of the instantaneous criteria before integration. We set

$$\check{\lambda} x(0)(S_0 - K) = \lambda_{ABM,DD}^{SAE} x^2(0) \sigma^2 S_0^2, \quad \check{\lambda} x(T)(S_T - K) = \lambda_{ABM,DD}^{SAE} x^2(T) \sigma^2 \mathbb{E}_0[S_T^2]$$

for the cases ABM and DD respectively (as per λ 's index).

Recalling that $x(0) = X$ and $x(T) = 0$, we have that the second equations (for T) are just identities, whereas the first ones lead to

$$\lambda_{ABM,DD}^{SAE} = \check{\lambda} \frac{S_0 - K}{X \sigma^2 S_0^2},$$

so that if we aim at comparing the SAE and the VaR or ES criteria outputs we may use the above λ in the risk criterion for SAE.

A second possible way to derive λ in SAE is to check what happens for the VWAP solution of the simpler case

$$x_0(t) = X \frac{T - t}{T}$$

and match the risk functions corresponding to this solution.

Straightforward calculations based on solving

$$\mathbb{E}_0 \left[\int_0^T \check{\lambda} x_0(t) (S_t - K) dt \right] = \mathbb{E}_0 \left[\int_0^T \lambda_{DD}^{SAE} x_0^2(t) \sigma^2 \mathbb{E}_0[S_t^2] dt \right]$$

in the DD case, while using the expansion $e^{\sigma^2 t} \approx 1 + \sigma^2 t$, leads to

$$\lambda_{DD}^{SAE} = \frac{\check{\lambda}}{2X\sigma^2} \frac{1}{(S_0 - K) \left(\frac{\sigma^2 T}{12} + \frac{S_0^2}{3(S_0 - K)^2} \right)}. \quad (6.1)$$

This is the formula we will use, and we will set also

$$\lambda_{ABM}^{SAE} := \lambda_{DD}^{SAE}. \quad (6.2)$$

6.1 Comparative Results

In this section we present a series of numerical results which compare the different models and risk criteria analysed in the paper. In order to stress the main features of each model we chose, in some instance, parameters which may be regarded as extreme or unrealistic. We shall look at optimal solution under a more realistic set of parameters, and in particular the execution horizon, later in the section.

In the graphs below, we compare the optimal solution of the following dynamics/risk criterion combinations: ABM+SAE, DD+SAE, GBM+VaR and DD+VaR.

The λ 's will be equalized through Equations (6.1) and (6.2).

In figure 1, the time horizon for the execution is set to one year, which is admittedly very long. The volatility of the asset is 20% and the DD shift K is 20% of the initial asset price. Most importantly the cost/risk parameter L is relatively low at 100 and the temporary market impact parameter η has been set to 1%. For the choice of parameters above, the optimal execution strategy is very similar in all the models and almost linear in t .

Figure 2 is based on the same parameters of figure 1 with the exception K which is equal to 80% of the initial asset price. Note that the instantaneous volatility of the DD is rescaled using formula (6.3) to ensure that the integrated volatility of the DD and GBM models are roughly of the same order of magnitude,

$$\sigma^{DD} = \sigma^{GBM} \frac{S_0}{S_0 - K}. \quad (6.3)$$

Since the paths generated under the GBM and DD are different, the two optimal solutions differ, as we show with a GBM path which having a stronger performance

than DD. The optimal execution policy under ABM and DD using the SAE risk function is not path dependent and it is virtually unchanged under the new parameter choice.

Figure 3 shows that under a combination of high volatility and high L , the optimal strategy under the VaR risk criteria can change sign, e.g. a selling policy may turn into a buying policy before T . This is due to the integral of S_u or Y_u in the optimal solution. When this term is relatively big compared to X then the amount of asset to execute $x(t)$ turns negative. Mathematically this is due to the lack of a smoothness constraint at 0 for the process $x(t)$.

Figure 4 shows how an increase in the market parameter η slows down the execution policy.

In figure 5, we reduced the time horizon to a more realistic 1 day. Even if we choose very high risk parameters, e.g. $L = 5000$, $\sigma = 99\%$ and $\eta = 0.1$, the optimal solution for all models is very close to linear.

Finally, note that the solution under the VaR risk function depends on the relative magnitude of X and the price of the asset. In order to illustrate this feature, in figure 6 we use the parameters of figure 3 but we increase the units to execute to 10000 (i.e. 100 times the initial asset price). Notice how the optimal solution becomes approximately linear regardless of the path. The solutions obtained using the SAE risk function are also linear because in the rescaling formula for the relevant λ parameter, the X appears in the denominator, making the risk contribution negligible.

Since in practical applications the execution horizon is rarely longer than a few days (more commonly it is measured in hours), the VWAP execution strategy will be a very close approximation of the optimal strategy obtained using more sophisticated models.

Remark 6.1. (Actual implementation and estimation of the asset dynamics). *In real world applications we observe only one realised path of the "true" asset dynamics and the solution of the two GBM/DD models will depend on the "calibrated" parameters $\sigma^{GBM} / (\sigma^{DD}, K)$. Since the path on which both models are to be estimated will be the same, the optimal solution of the two models will be much closer than in the example above where the paths differ (note however that the same brownian shock process W realization has been used to sample both DD and GBM). If we had to be really precise, we would have to simulate a path using a third, more realistic model, and then calibrate the GBM and DD parameters to such path. At that point we would apply the optimal solution formula for the 2 models and compare the two solutions. Although the calibration of the GBM to historical data is*

quite simple, once the drift is assumed to be zero, calibrating the DD model (and K in particular) is slightly more complicated, and may account for higher order moments.

This method would be more ideal, but we may anticipate that results would be even closer than in our examples above, showing probably even smaller departures from the trivial VWAP solution.

An alternative calibration to fit the parameters could be an option pricing calibration, obtaining the dynamics statistics under the pricing measure, rather than the historical one, but for the optimal trade execution problem we need to be careful about mixing historical measure and risk neutral measure statistics.

7 Conclusions

In this paper we solved the optimal trade execution problem under the Value-at-risk or Expected shortfall based criteria of Gatheral and Schied when the underlying unaffected stock price follows a displaced diffusion model, encompassing arithmetic and geometric Brownian motion type features at the same time. We further introduced an alternative risk criterion called Squared-Asset Expectation, and we noticed that the optimal trade execution strategy does not depend strongly on the specific risk criterion one adopts. We find that the optimal solution is close to the basic linear Volume Weighted Average Price (VWAP) solution regardless of the specific diffusion dynamics or risk criterion that is chosen, especially on realistic trading horizons of a few days or hours, and unless unrealistically large values of the risk or dynamics parameters are taken. Our preliminary findings suggest the conjecture that more general dynamics, beyond simple diffusions, need to be considered to see a relevant impact on the optimal strategy. A similar conclusion holds also for the risk criterion.

References

- [1] Almgren, R., Chriss, N. (1999). Value under liquidation. Risk, Dec. 1999.
- [2] Almgren, R., Chriss, N. (2000). Optimal execution of portfolio transactions. J. Risk 3, 5-39 (2000).

- [3] Gatheral, J., and Schied, A. (2011). Optimal Trade Execution under Geometric Brownian Motion in the Almgren and Chriss Framework. *International Journal of Theoretical and Applied Finance*, Vol. 14, No. 3, pp. 353–368
- [4] Forsyth, P. (2009). A Hamilton Jacobi Bellman approach to optimal trade execution. Preprint, available at http://www.cs.uwaterloo.ca/~paforsyt/optimal_trade.pdf
- [5] Forsyth, P., Kennedy J., Tse T.S., Windcliff H. (2009). Optimal Trade Execution: A Mean-Quadratic-Variation Approach. Preprint (2009) available at http://www.cs.uwaterloo.ca/~paforsyt/quad_trade.pdf
- [6] Schied, A. (2012). Robust Strategies for Optimal Order Execution in the Almgren–Chriss Framework. Working Paper.

Appendices

A Derivation of the ODE for the Optimal Strategy

The optimisation problem (5.3) requires that we find the path $x(t)$ which minimises the loss function

$$L[x] \equiv \mathbb{E} \left[\int_0^T \dot{x}(t)^2 dt + L\lambda \int_0^T \sigma^2 x^2(t) g(t) dt \right]. \quad (\text{A.1})$$

In order to do so, consider the following perturbation of the processes $x(t)$ and $\dot{x}(t)$

$$\begin{aligned} x^\epsilon(t) &= x(t) + \epsilon h(t) \\ \dot{x}^\epsilon(t) &= \dot{x}(t) + \epsilon \dot{h}(t) \end{aligned}$$

where $h(t)$ is an arbitrary function satisfying $h(0) = h(T) = 0$ and ϵ is a real constant. Substituting the perturbed path into (A.1) we obtain

$$H(\epsilon) = \mathbb{E} \left[\int_0^T (\dot{x}^\epsilon(t))^2 dt + L\lambda \int_0^T \sigma^2 (x^\epsilon(t))^2 g(t) dt \right].$$

Taking the first derivative with respect to ϵ and evaluating the resulting expression at $\epsilon = 0$, it follows that

$$\dot{H}(0) = \mathbb{E} \left[2 \int_0^T \dot{x}(t) \dot{h}(t) dt + 2L\lambda \int_0^T \sigma^2 x(t) h(t) g(t) dt \right]. \quad (\text{A.2})$$

Using the integration by parts formula and the constraint $h(0) = h(T) = 0$, we obtain the equality

$$\int_0^T \dot{x}(t) \dot{h}(t) dt = - \int_0^T \ddot{x}(t) h(t) dt,$$

which allows us to simplify (A.2) as follows

$$\dot{H}(0) = \mathbb{E} \left[2 \int_0^T (\ddot{x}(t) - L\lambda \sigma^2 x(t) g(t)) h(t) dt \right].$$

The optimal path $x(t)$ is obtained by setting $\dot{H}(0) = 0$. Since the function $h(t)$ was chosen arbitrarily, the following must hold for all $0 \leq t \leq T$

$$\ddot{x}(t) - L\lambda \sigma^2 g(t) x(t) = 0.$$

B Numerical algorithm for optimal solutions

Consider the ODE

$$\ddot{x}(t) = k^2 g(t) x(t).$$

Simple manipulations show that $g(t)$ can be written as follows

$$g(t) = Y_0^2 \left[\left(\frac{K}{Y_0} \right)^2 + 2 \left(\frac{K}{Y_0} \right) + \exp(\sigma^2 t) \right] \quad (\text{B.1})$$

$$= \bar{k}^2 (\alpha_0 + e^{\sigma^2 t}) \quad (\text{B.2})$$

where

$$\bar{k}^2 \equiv (Y_0 k)^2$$

and

$$\alpha_0 \equiv \left(\frac{K}{Y_0} \right)^2 + 2 \frac{K}{Y_0}.$$

Assume that the solution of the ODE can be expanded in a Taylor series around $t = 0$

$$x(t) = \sum_{j=0}^{\infty} a_j t^j. \quad (\text{B.3})$$

The coefficient of the series can be found by substituting (B.3) into the ODE,

$$\sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} t^j = \bar{k}^2 (\alpha_0 + \exp(\sigma^2 t)) \left(\sum_{j=0}^{\infty} a_j t^j \right).$$

Substituting the exponential term with its Taylor expansion in t and exchanging the order of summation we obtain

$$\begin{aligned} \sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} t^j &= \bar{k}^2 (1 + \alpha_0) \left(\sum_{j=0}^{\infty} a_j t^j \right) + \bar{k}^2 \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \frac{\sigma^i}{i!} a_j t^{j+i} \\ &= \bar{k}^2 (1 + \alpha_0) \left(\sum_{j=0}^{\infty} a_j t^j \right) + \bar{k}^2 \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \frac{\sigma^i}{i!} a_{j-i} t^j. \end{aligned} \quad (\text{B.4})$$

Truncating both sums at m and equating the coefficient of the same order in t on both sides of the equation we obtain a recursive expression for the coefficients a_j ,

$$a_{j+2} = \frac{\bar{k}^2}{(j+2)(j+1)} \left((1 + \alpha_0)a_j + \sum_{k=1}^j a_{j-k} \frac{v^k}{k!} \right).$$

for $j = 0$ to $m - 2$.

The expression above can be written in a more compact form using vector notation

$$a_{j+2} = \frac{\bar{k}^2}{(j+2)(j+1)} ((1 + \alpha_0)a_j, A_{j-1}) \cdot V_j$$

where we have defined

$$A_j \equiv (a_0, a_1, \dots, a_j)'$$

$$V_j \equiv \left(1, v, \frac{v^2}{2!}, \dots, \frac{v^j}{j!} \right)'$$

Using the initial condition $x(0) = X$ it follows that $a_0 = X$. On the other hand, a_1 can be derived numerically via a simple root search using the terminal condition $x(T) = 0$.

Figure 1: Model Comparison: $S_0 = 100$, $X = 100$, $T = 1$, $K = 20$, $\sigma = 0.2$, $\eta = 1$, $L = 100$

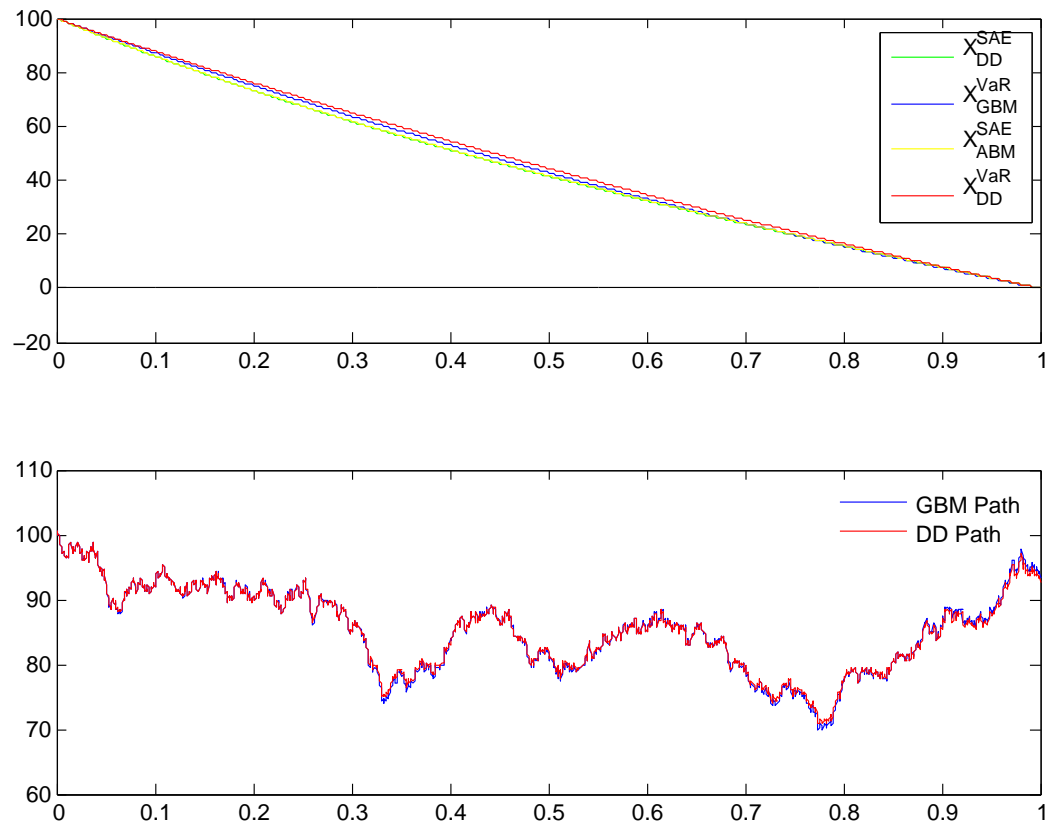


Figure 2: Model Comparison: $S_0 = 100$, $X = 100$, $T = 1$, $K = 80$, $\sigma = 0.2$, $\eta = 1$, $L = 100$

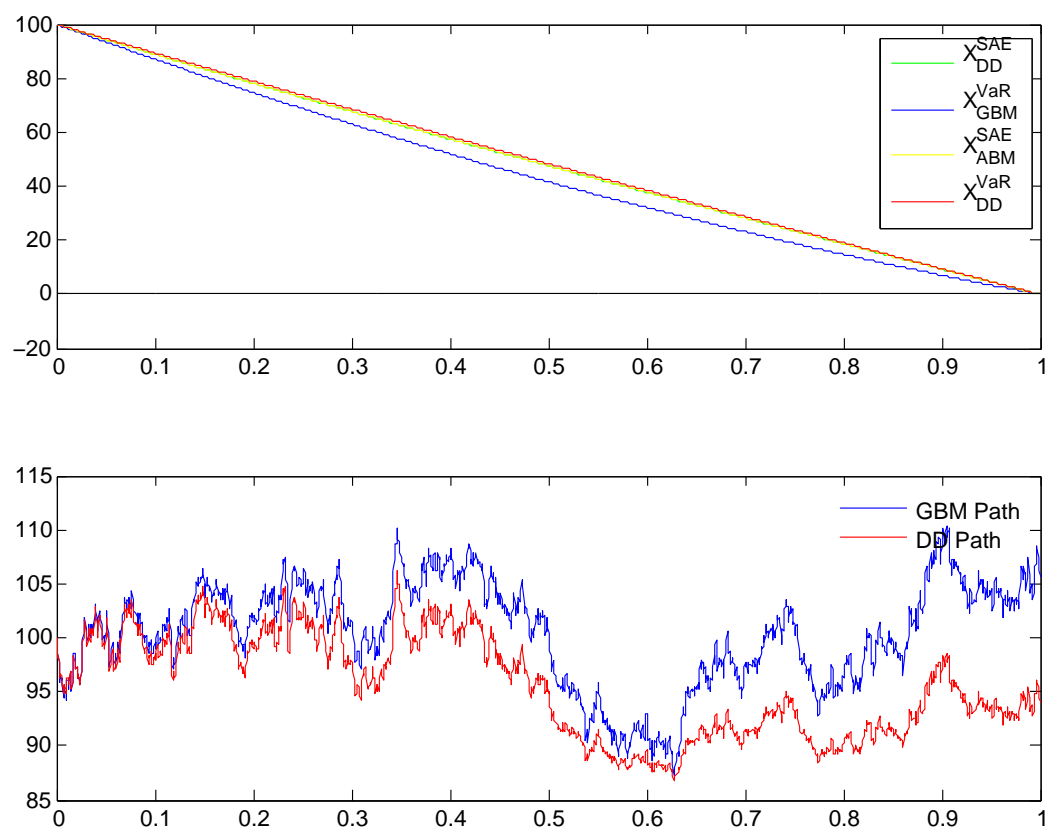


Figure 3: Model Comparison: $S_0 = 100$, $X = 100$, $T = 1$, $K = 20$, $\sigma = 0.75$, $\eta = 1$, $L = 500$

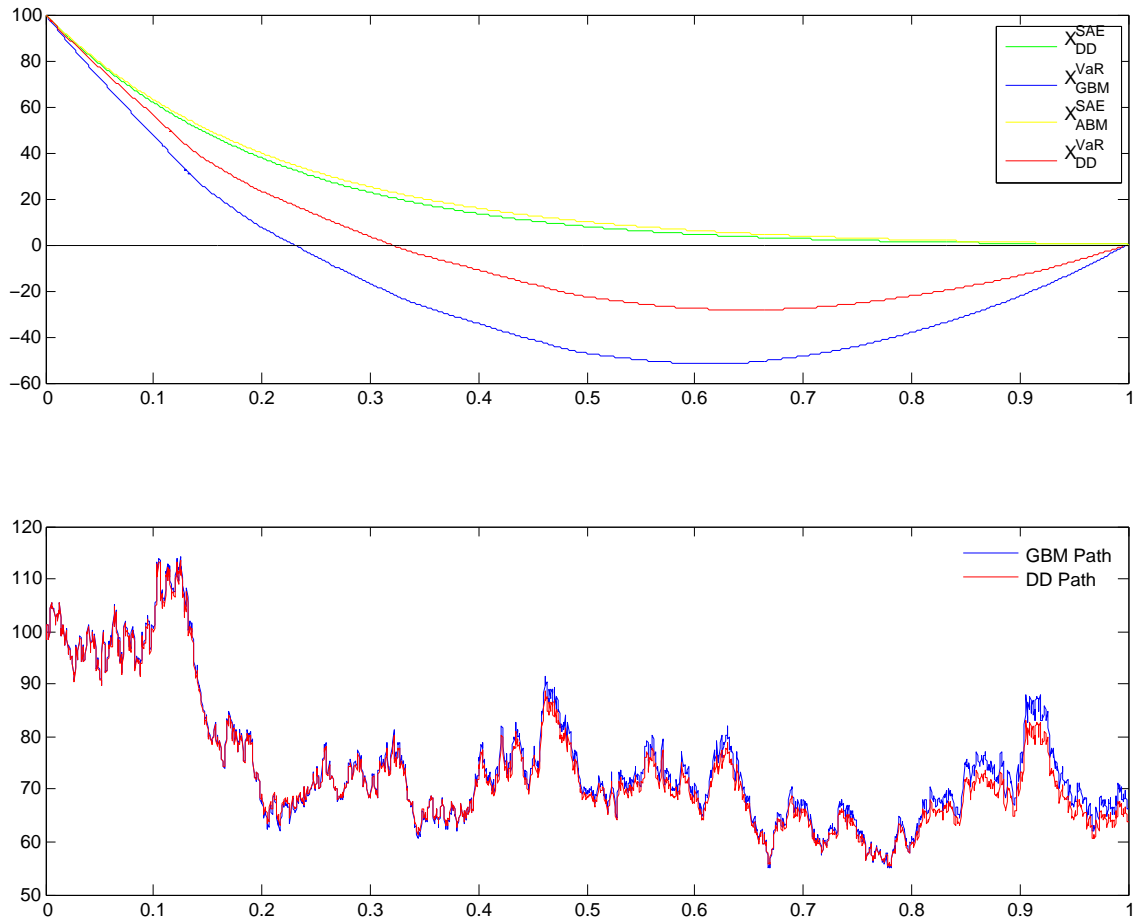


Figure 4: Model Comparison: $S_0 = 100$, $X = 100$, $T = 1$, $K = 20$, $\sigma = 0.75$, $\eta = 5$, $L = 500$

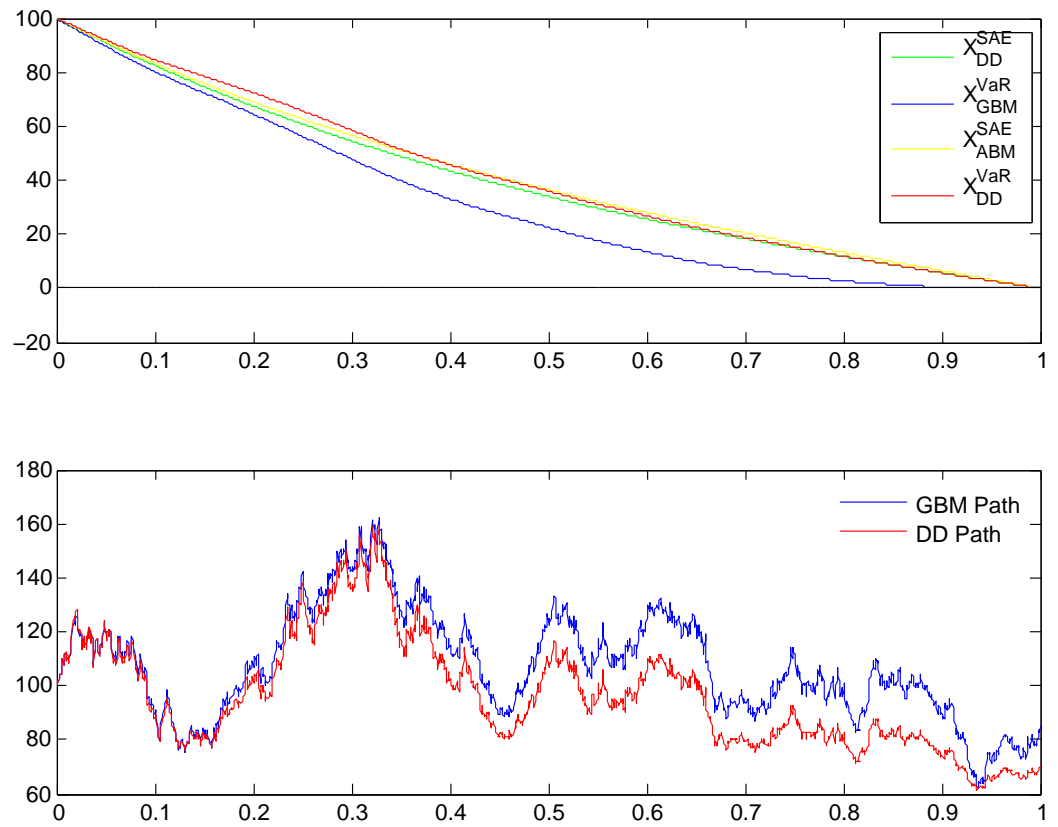


Figure 5: Model Comparison: $S_0 = 100$, $X = 100$, $T = 1/250$, $K = 90$, $\sigma = 0.99$, $\eta = 0.1$, $L = 5000$

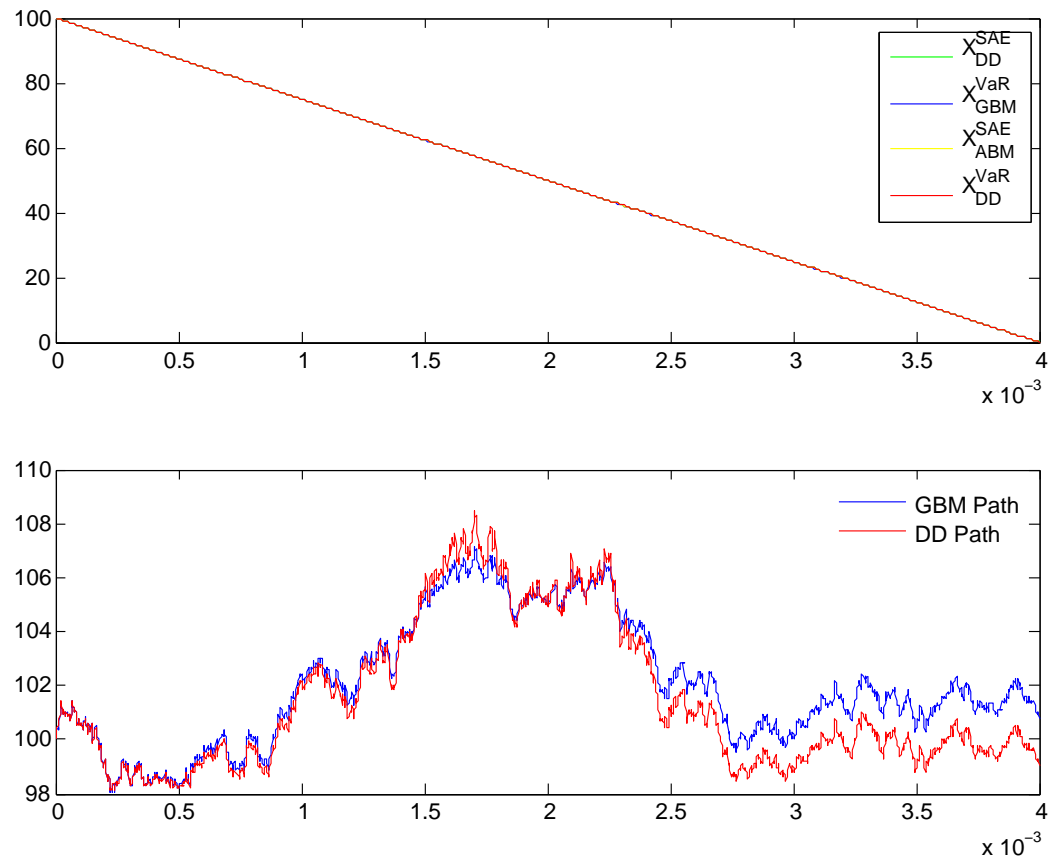


Figure 6: Model Comparison: $S_0 = 100$, $X = 10000$, $T = 1$, $K = 20$, $\sigma = 0.75$, $\eta = 1$, $L = 500$

